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LETTER TO THE EDITOR

On spontaneous symmetry breakdown in dynamical systems

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Abstract. We describe some implications of spontaneous symmetry breaking (SSB) in the context of far-from-equilibrium transitions. We show that the SSB of continuous symmetries lead to the appearance of poles in the Fourier transforms of response functions in complete analogy with the Goldstone theorem in Hamiltonian systems.

The concept of spontaneous symmetry breakdown (SSB) has proved fruitful for the description of a wide range of phenomena. First described in the framework of phase transition, it has been applied successfully in the context of quantum field theory. Both the spontaneous magnetization of ferromagnetic systems below the Curie temperature and the dynamics of the π-mesons at low energies (as compared to the GeV scale) are thought to represent the spontaneous breakdown of given symmetries. In such cases, it is often stated that SSB occurs when the lowest energy state of a given system does not share all the symmetries of the Hamiltonian. The very notion of symmetry breaking has, however, been extended to a much larger class of phenomena. Far from equilibrium, a system may undergo various bifurcations leading to stable states characterized by given symmetries [1]. Fixed points, limit cycles and pattern formations have been observed both experimentally and on computer simulations. It is usually assumed that these far-from-equilibrium transitions (FFET) can be properly described by generalized Landau–Ginsburg models which, as a rule, cannot be cast in Hamiltonian form [1].

A well known feature of the SSB of a continuous symmetry is the appearance of the so-called 'Goldstone modes' [2]. They originate in the (continuous) degeneracy of the ground state left by the symmetry breakdown. In the context of FFET, they correspond to the concept of 'phase dynamics' (see e.g. [3]). In this letter, we would like to point out that the validity of the 'Goldstone theorem' is not restricted to Hamiltonian systems. More precisely, we consider a class of stochastically perturbed dynamical systems which, by hypothesis, lead to non-trivial solutions for the mean value of the vector field. The 'theorem' states, in this case, that each broken generator of a continuous symmetry will lead to a simple pole in the Fourier transform of a corresponding response function.

Consider the following stochastically perturbed dynamical system

$$d_t X_i = F_i(X) + W_i \qquad i = 1, \ldots, N \tag{1}$$

where the F_i are analytical functions of the X_i and the W_i are Gaussian white noises:

$$\langle W_i \rangle = 0 \qquad \langle W_i(t) W_j(s) \rangle = \Gamma \delta_{ij} \delta(t-s).$$

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The system is meant to be invariant under SO(N) transformations, i.e. formally

$$[d_t, \bar{\delta}] = 0$$

for

$$\bar{\delta}^{kl} X_i = (T^{kl})_{ij} X_i$$

where the T's represent the SO(N) generators: $(T^{kl})_{ij} = (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k)$. The generating functional for the Green functions is given by [4]

$$Z[j, \hat{j}] = \int \mathrm{d}X\, \mathrm{d}\hat{X}\, J[X] \exp\left\{ \int \mathrm{d}t(-L + j \cdot X + \hat{j} \cdot \hat{X}) \right\} \tag{2}$$

where

$$L = \tfrac{1}{2}\Gamma |\hat{X}|^2 - \mathrm{i}\hat{X} \cdot (d_t X - F).$$

The hatted field is a mere Lagrange multiplier introduced to generate the response functions.

The Jacobian $J[X] = |\delta W / \delta X|$ can be most conveniently represented by anticommuting Grassmann variables [5]. From (2), the following 'Schwinger equation' can be derived:

$$\left[\Gamma \frac{\delta}{\delta \hat{j}_i(t)} - \mathrm{i}\left[d_t \frac{\delta}{\delta j_i(t)} - F_i\left(\frac{\delta}{\delta j(t)} \right) \right] - \hat{j}_i(t) \right] Z[\hat{j}, j] = 0 \tag{3}$$

which leads to

$$d_t \bar{X}_i = \bar{F}_i$$

where we have set

$$\langle X_i \rangle = \bar{X}_i \qquad \langle F_i \rangle = \bar{F}_i$$

and

$$d_t \langle \hat{X}_i(t) X_j(s) \rangle - \langle \hat{X}_i(t) F_j(s) \rangle = \mathrm{i}\delta_{ij}\delta(t-s).$$

The Fourier transform of the latter is given by

$$[\mathrm{i}\omega\delta_{ij} - \tilde{\pi}_{ij}(\omega)] G_{jk}(\omega) = \mathrm{i}\delta_{jk} \tag{4}$$

where $G(\omega)$ is the Fourier transform of the first response function and, by definition,

$$\frac{\delta \bar{F}_i(t)}{\delta \bar{X}_j(s)} = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega(t-s)} \tilde{\pi}_{ij}(\omega).$$

The SO(N) symmetry can be represented by the following Ward identity [6]:

$$\mathrm{Tr}\left[\bar{\delta}^{\mathrm{t}} \boldsymbol{j} \frac{\delta}{\partial \boldsymbol{j}} \right] Z[\boldsymbol{j}] = 0 \qquad \boldsymbol{j} = (\hat{j}, j) \tag{5}$$

where the superscript t denotes the transposed transformation. From (5), one can derive the following equation:

$$\int \frac{\mathrm{d}\omega}{2\pi} [\mathrm{i}\omega\delta_{ij} - \tilde{\pi}_{ij}(\omega)] \bar{\delta}\tilde{\bar{X}}_j(\omega) = 0 \tag{6}$$

where $\tilde{\bar{X}}$ is the Fourier transform of $\bar{X}$.

In the simplest case, the SO(N) symmetry will be spontaneously broken when

$$\bar{F}_i = 0 \qquad i = 1, \ldots, N$$

while $\tilde{\bar{X}}(\omega) = \delta(\omega)\delta_{iI} \times \text{constant}$ for some I.

Then, from (6), one must have

$$\tilde{\pi}_{ik}(0) = 0 \qquad \forall k \neq I$$

so that (4) implies

$$\lim_{\omega \to 0} \omega G_{ii}(\omega) = 1 \qquad \forall i \neq I \text{ (no summation on } i).$$

The 'Goldstone modes' X_i, $i \neq I$, have a pole in their first response function. In the same way, the corresponding correlation functions will show a double pole in ω. The situation is a little more complicated when time-symmetry breaking occurs. The system might bifurcate to a stable limit cycle, for instance. The simplest time-dependent solutions that conserve the norm are of the following type:

$$\bar{F}_i(t) = \Omega(T^{IK})_{ij}\bar{X}_j(t) \qquad \text{for some } I, K.$$

That is to say

$$(d_t\delta_{ij} - \Omega(T^{IK})_{ij})\bar{X}_j(t) = 0 \qquad d_t|\bar{X}| = 0.$$

It is seen that, by performing a *local* SO(2) transformation, one can reduce this case to the previous one. In more general cases, space-dependent structure might appear as a result of a FFET. These patterns are often described by adding a diffusion term to (1), leading to a reaction-diffusion system [1]:

$$(\partial_t - D\nabla^2)X_i = F_i(X) + W_i$$

where again, the W's are δ-correlated white noises. Equations (4) and (6) are replaced by

$$[(\mathrm{i}\omega + D\boldsymbol{q}^2)\delta_{ij} - \tilde{\pi}_{ij}(\boldsymbol{q}, \omega)]G_{jk}(\boldsymbol{q}, \omega) = \mathrm{i}\delta_{jk}$$

$$\int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d}[(\mathrm{i}\omega + D\boldsymbol{q}^2)\delta_{ij} - \tilde{\pi}_{ij}(\boldsymbol{q}, \omega)]\bar{\delta}\tilde{\bar{X}}_j(\boldsymbol{q}, \omega) = 0$$

where $(\boldsymbol{q}, \omega)$ are the associated Fourier variables of $(\boldsymbol{x}, t)$. As above, in the simplest case,

$$\tilde{\bar{X}}(\boldsymbol{q}, \omega) = \delta(\omega)\delta^d(\boldsymbol{q})\delta_{iI} \times \text{constant for some } I.$$

This implies

$$\lim_{(\boldsymbol{q},\omega)\to 0} -\mathrm{i}(\mathrm{i}\omega + D\boldsymbol{q}^2)G_{ii}(\boldsymbol{q}, \omega) = 1 \qquad \forall i \neq I \text{ (no summation on } i).$$

In a similar way as above, the simplest spacetime dependent solutions that conserve the norm are of the form:

$$(\partial_t\delta_{ij} - \Omega(T^{IK})_{ij})\bar{X}_j = 0$$

$$(\nabla\delta_{ij} - \boldsymbol{k}(T^{I'K'})_{ij})\bar{X}_j = 0$$

$$\partial_t|\bar{X}| = \nabla|\bar{X}| = 0.$$

Here again, local gauge transformations in $(\boldsymbol{x}, t)$ will reduce this case to the previous one. We have thus shown that 'simple' FFETs, in particular the ones that conserve the norm, correspond to SSB similar to those described in the context of equilibrium transitions. In particular, the SSB of a continuous symmetry lead to the appearance of poles in the Fourier transform of response functions.

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